CIMPA Summer School 2014
Random structures, analytic and probabilistic approaches
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Pólya urn models: exercises

1 Exercises

Exercise 1.
When $R = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$, code and count all histories of length 2 leading from $\binom{2}{0}$ to $\binom{4}{4}$.

Exercise 2.
Whenever $R = SI_2$, compute all numbers H_n, $n \geq 0$. [This urn is Pólya's original one in his article published in 1930.]

Exercise 3.
For any urn, if $N = \alpha + \beta$, show that the total number of histories of length n starting from $\binom{\alpha}{\beta}$ equals $N\big(N+S\big)\big(N+2S\big)\dots\big(N+(n-1)S\big) = n!S^n \binom{\frac{N}{S}+n-1}{n}$.

Exercise 4. For any urn (*i.e.* for any R), show that

$$H\left(1,1,z \,\middle|\, \begin{array}{c} u_0 \\ v_0 \end{array}\right) = \left(\frac{1}{1-Sz}\right)^{\frac{u_0+v_0}{S}}.$$

Exercise 5.
For the original urn ($R = SI_2$),

$$H\left(x,y,z \,\middle|\, \begin{array}{c} u_0 \\ v_0 \end{array}\right) = \frac{x^{u_0}y^{v_0}}{\left(1-Szx^S\right)^{\frac{u_0}{S}}\left(1-Szy^S\right)^{\frac{v_0}{S}}}.$$

Exercise 6.

All notations are those of Section 3.

6.1- (Linear functions)
Show that if V is a real vector space and if $f : \mathbb{R}^2 \to V$ is linear, then

$$\Phi(f) = f \circ A$$

where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $A(v) = A\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} := {}^tR\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}$.

6.2- (Vector-valued martingale)

Denote $\tau := \alpha + \beta$. Show that the process $\left(\gamma_{n,\tau}\left({}^tR\right)^{-1}(U_n)\right)_n$ is a martingale (with regard to the natural filtration) as soon as it is defined, *i.e.* as soon as all matrices $I_2 + \frac{1}{kS+\tau}R$, $k \in \mathbb{N}$ are invertible. Show that this martingale is not defined if, and only if $m \leq -1$ and S divides $m + \alpha + \beta$.

6.3- (Expectation)

Let u_1 and u_2 be the eigenforms defined in Section 1. Verify (or remember!) that $u_1 \circ A = Su_1$ and $u_2 \circ A = mu_2$. Show that for any $n \in \mathbb{N}$,

$$\mathbf{E}u_1(U_n) = n + \frac{\tau}{S}$$

and, when n tends to infinity,

$$\mathbf{E}u_2(U_n) = \frac{\Gamma\left(\frac{\tau}{S}\right)}{\Gamma\left(\frac{\tau}{S}+\sigma\right)} \frac{b\alpha - c\beta}{S} n^{\sigma}\left(1 + O\left(\frac{1}{n}\right)\right).$$

When $R \neq SI_2$, using that $v = u_1(v)v_1 + u_2(v)v_2$ for any vector $v \in \mathbb{R}^2$, show that, when n tends to infinity,

$$\mathbf{E}U_n \sim nv_1$$

Example of 2-3 trees.

6.4- (Real-valued projected martingales)

Show that

$$\left(\frac{u_1(U_n)}{nS+\tau}\right)_n$$

is an almost surely bounded (thus convergent) martingale and compute its expectation. Show that

$$\left(\frac{u_2(U_n)}{\gamma_{n,\tau}(m)}\right)_n$$

is a martingale as well, as soon as $m \geq 0$ or $m + \tau$ is not a multiple of S.

6.5- (Second moments)

Denote by P and Q the 2-variable polynomials defined by

$$P(x,y) = u_1(x,y)\Big(u_1(x,y)+1\Big) \quad \text{and} \quad Q(x,y) = \Big(u_1(x,y)+\sigma\Big)u_2(x,y).$$

Show that $\Phi(P) = 2SP$ and $\Phi(Q) = (S+m)Q$ and prove the asymptotics when n tends to infinity

$$\mathbf{E}P(U_n) = n^2\left(1 + O\left(\frac{1}{n}\right)\right)$$

and

$$\mathbf{E}Q(U_n) = \frac{\Gamma\left(\frac{\tau}{S}\right)}{\Gamma\left(\frac{\tau}{S}+\sigma\right)} \frac{b\alpha - c\beta}{S} n^{1+\sigma}\left(1 + O\left(\frac{1}{n}\right)\right)$$

(if one feels depressed, one can just show that $Q\left(U_n\right) \in O\left(n^{1+\sigma}\right)$, :-)).

Suppose that $\sigma \neq 1/2$ and denote

$$R = u_2^2 - \frac{bc\sigma^2}{1-2\sigma}u_1 + (b-c)\,\sigma u_2.$$

Show that, in this case, $\Phi\left(R\right) = 2mR$ and that, when n tends to infinity,

$$\mathbf{E}R\left(U_n\right) = \frac{\Gamma\left(\frac{\tau}{S}\right)}{\Gamma\left(\frac{\tau}{S}+2\sigma\right)}R\left(\alpha,\beta\right)n^{2\sigma}\left(1+O\left(\frac{1}{n}\right)\right).$$

Show that $(1, u_1, u_2, P, Q, R)$ is a basis of the vector space $\mathbb{R}_2[x, y]$ of polynomials of degree less than or equal to 2. Write x^2, xy and y^2 in this basis and compute the asymptotics of the co-moment matrix $\mathbf{E}\left[U_n{}^tU_n\right]$ and of the covariance matrix $\mathbf{E}\left[\left(U_n - \mathbf{E}U_n\right)^t\left(U_n - \mathbf{E}U_n\right)\right]$ (one has to discuss whether $\sigma < 1/2$ or $\sigma > 1/2$).

Check what happens when $\sigma = 1/2$ and do the same job using $T = u_2^2 + \frac{2b-m}{2}u_2$ instead of R.

6.6- (For large urns, the second projected martingale is square-bounded)
Suppose that $\sigma > 1/2$. Expressing u_2^2 as a function of R, u_1 and u_2, show that the martingale $\left(\frac{u_2(U_n)}{\gamma_{n,\tau}(m)}\right)_n$ is bounded in L^2, thus convergent.

Exercise 7 (triangular urn).
Assume that $b = 0$, so that $R = \begin{pmatrix} S & 0 \\ S-m & m \end{pmatrix}$. Assume also that the initial number of black balls is non zero, *i.e.* that $\beta \neq 0$ (and check that $\beta = 0$ leads to a degenerate process). Let as above u_1 and u_2 be the linear forms

$$u_1(x,y) = \frac{x+y}{S} \quad\text{and}\quad u_2(x,y) = \frac{y}{S}.$$

For any $p \in \mathbb{N}^*$, let also A_p and B_p be the bivariate polynomials

$$A_p = u_1\left(u_1+1\right)\dots\left(u_1+p-1\right) = \frac{\Gamma\left(u_1+p\right)}{\Gamma\left(u_1\right)}$$

and

$$B_p = u_2\Big(u_2+\sigma\Big)\dots\Big(u_2+(p-1)\sigma\Big) = \frac{\Gamma\left(u_2+p\sigma\right)}{\Gamma\left(u_2\right)}.$$

Show that $\Phi\left(A_p\right) = pSA_p$ (as always, even if R is not triangular) and that $\Phi\left(B_p\right) = pmB_p$ for any $p \geq 1$. Deduce from this that, when n tends to infinity,

$$\mathbf{E}B_p\left(U_n\right) = \frac{\Gamma\left(\frac{\tau}{S}\right)}{\Gamma\left(\frac{\tau}{S}+p\sigma\right)}\frac{\Gamma\left(\frac{\beta}{S}+p\sigma\right)}{\Gamma\left(\frac{\beta}{S}\right)}n^{p\sigma}\left(1+O\left(\frac{1}{n}\right)\right).$$

- Assume that $m \geq 1$.

Using the inversion formula

$$u_2^p = \sum_{k=1}^{p} (-\sigma)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} B_k,$$

show that, for any $p \geq 1$,

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{u_2\left(U_n\right)}{n^\sigma}\right)^p = \frac{\Gamma\left(\frac{\tau}{S}\right)}{\Gamma\left(\frac{\tau}{S}+p\sigma\right)} \frac{\Gamma\left(\frac{\beta}{S}+p\sigma\right)}{\Gamma\left(\frac{\beta}{S}\right)}. \tag{1}$$

so that the number of black balls $U_n^{(2)} = Su_2\left(U_n\right)$ converges in law to a random variable having the right side of Equality (1) as p-th moment (to make a complete proof of that fact, one has to check that a distribution having such a p-th moment is determined by its moments, which can be done by computing the asymptotics of (1) as p tends to infinity with the help of Stirling Formula). This law can be related to stable laws or to Mittag-Leffler ones.

• Assume that $m = 0$. Show that the process is deterministic (degenerate case).

• Assume that $m \leq -1$. Show that the number of black balls tends almost surely to zero (degenerate case again).

2 Hints

Exercise 1.
One possible solution: start from $W_0 = \mathtt{r}^2$. One can draw a tree of all possibilities: $W_1 \in \{\mathtt{rb}^3\mathtt{r}, \mathtt{r}^2\mathtt{b}^3\}$, then $W_2 \in \{\mathtt{rn}^6\mathtt{r}, \mathtt{r}^3\mathtt{n}^4\mathtt{r}, \mathtt{rnr}^2\mathtt{n}^3\mathtt{r}, \mathtt{rn}^2\mathtt{r}^2\mathtt{n}^2\mathtt{r}, \mathtt{rn}^3\mathtt{rn}^3\}$ ou $W_2 \in \{\mathtt{rb}^3\mathtt{rb}^3, \mathtt{r}^2\mathtt{b}^6, \mathtt{r}^4\mathtt{b}^4, \mathtt{r}^2\mathtt{br}^2\mathtt{b}^3, \mathtt{r}^2\mathtt{b}^2\mathtt{r}^2\mathtt{b}^2\}$. Amongst the ten histories of length 2, six of them lead to $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and four lead to $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$: starting from two red balls, the probability that the urn contains four red balls and four black ones after two drawings is $3/5$.

Beware: in the example, the configuration $\mathtt{rn}^3\mathtt{rn}^3$ is reached by two different histories. We count histories, not the different word that are potentially obtained.

Exercise 2
This is elementary enumerative combinatorics. Make the picture of a path in $\mathbb{N}^2$ and count the histories that follow each of these paths. For any $(p,q) \in \mathbb{N}^2$ such that $p+q=n$, one gets

$$\begin{aligned} H_n\left(\begin{array}{cc} \alpha & \alpha+pS \\ \beta & \beta+qS \end{array}\right) &= \binom{n}{p} \alpha\big(\alpha+S\big)\dots\big(\alpha+(p-1)S\big)\beta\big(\beta+S\big)\dots\big(\beta+(q-1)S\big) \\ &= n!S^n \binom{\frac{\alpha}{S}+p-1}{p} \binom{\frac{\beta}{S}+q-1}{q} \ ; \end{aligned}$$

all others H_n vanish.

Exercise 5.
Computations on multivariate power series, based on the formula $\frac{1}{(1-X)^N} = \sum_{n\geq 0} \binom{N+n-1}{n} X^n$.

Exercise 6.

6.2- Apply **6.1-** to $f = \mathrm{Id}$. This leads to $\mathbf{E}\left(U_{n+1}\middle|U_n\right) = \left(I_2 + \frac{1}{nS+\tau}A\right)(U_n)$. This implies all the answers, because A is diagonalizable, with eigenvalues S and m.

6.3- An induction using **6.1-** leads to $\mathbf{E}u_1(U_n) = \gamma_{n,\tau}(S) \times u_1(U_0) = \frac{nS+\tau}{\tau} \times \frac{\tau}{S}$. For u_2, apply Stirling Formula to $\mathbf{E}u_2(U_n) = \gamma_{n,\tau}(m) \times u_2(U_0)$ with a O-remainder. The third assertion is obtained by addition of asymptotic developments.

6.4- Using **6.1-** again, one gets $\mathbf{E}\left(u_1(U_{n+1})\middle|U_n\right) = \left(1 + \frac{S}{nS+\tau}\right) \times u_1(U_n)$, so that $\mathbf{E}\left(\frac{u_1(U_{n+1})}{(n+1)S+\tau}\middle|U_n\right) = \frac{u_1(U_n)}{nS+\tau}$, proving the martingale property. Same argument from $\mathbf{E}\left(u_2(U_{n+1})\middle|U_n\right) = \left(1 + \frac{m}{nS+\tau}\right) \times u_2(U_n)$.

6.5- One gets $\Phi(P)$, $\Phi(Q)$ and $\Phi(R)$ by simple computation. Since $\Phi(P) = 2SP$, $\mathbf{E}P(U_n) = \gamma_{n,\tau}(2S) \times P(U_0)$ and the required asymptotics for $\mathbf{E}P(U_n)$ is obtained thanks to Stirling Formula. *Idem* for $\mathbf{E}Q(U_n)$ and $\mathbf{E}R(U_n)$. The remainder of the exercise is completely left to the reader.

6.6- $u_2^2 = R + \frac{bc\sigma^2}{1-2\sigma}u_1 - (b-c)\sigma u_2$, so that $\mathbf{E}u_2^2(U_n) = c_1 n^{2\sigma}(1 + O(1/n)) + c_2 n + c_3 n^{\sigma}(1 + O(1/n))$ where c_1, c_2 and c_3 are constants. Since $\sigma > 1/2$, the principal term is the one in $n^{2\sigma}$, proving that the martingale is square bounded (use Stirling formula again to get the asymptotics of $\gamma_{n,\tau}(m)^2$).