Enumeration of planar graphs: how symbolic method and singularity analysis apply

Nicolas Pouyanne
Laboratoire de mathématiques de Versailles
UVSQ, CNRS, Université Paris-Saclay, France

Conference *Mathematics and random structures*
27 to 30 August 2018
Birzeit University, Palestine

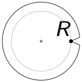
Transfer Theorem: a simple version

Let $S(x) = \sum s_n x^n$ be a power series. Assume that:
(i) S has $R > 0$ as a convergence radius
(ii) in a camembert♫ *domain around R, the function $x \mapsto S(x)$ is analytic and admits an expansion of the form*

$$S(x) \sim C \left(1 - \frac{x}{R}\right)^{\alpha}$$

when x tends to R in the domain, with $C \in \mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geqslant 0}$.
Then, when n tends to infinity,

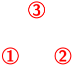
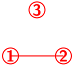
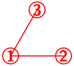
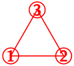
$$s_n \sim \frac{C}{\Gamma(-\alpha)} R^{-n} n^{-\alpha-1}.$$

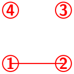
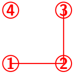
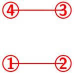
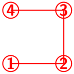
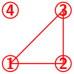
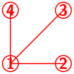
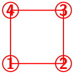
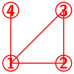
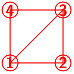
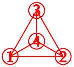
♫A *camembert domain around R*: an open set 

Small labelled planar graphs

1×  $g_1 = 1$

1×  and 1×  $g_2 = 2$

1×  and 3×  and 3×  and 1×  $g_3 = 8$

1×  and 6×  and 12×  and 3× 
and 12×  and 4×  and 4×  and 3× 
and 12×  and 6×  and 1×  $g_4 = 64$

$g_5 = 1023, \ldots$

Claim: reaching $B_2(x, y)$

There are bivariate power series $U(x,z)$, $D(x,y)$ and $M(x,y)$ related by the following relations:

$$U(x,y) = xy\left(1 + y\,(1+U(x,y))^2\right)^2 \tag{1}$$

$$M(x,y) = Rat\,(x,y,U(x,y)) \tag{2}$$

$$\frac{M\,(x,D(x,y))}{2x^2D(x,y)} - \log\frac{1+D(x,y)}{1+y} + \frac{xD^2(x,y)}{1+xD(x,y)} = 0 \tag{3}$$

$$\frac{\partial B_2(x,y)}{\partial y} = \frac{x^2}{2}\left[\frac{1+D(x,y)}{1+y} - 1\right]. \tag{4}$$

In Formula (2), *Rat* denotes an explicit simple rational fraction in 3 variables on $\mathbb{Q}$ which can be easily written on one line.

A very short bibliography

P. Flajolet and R. Sedgewick,
Analytic Combinatorics,
Cambridge University Press, 2009.

Free acces at
http://algo.inria.fr/flajolet/Publications/book.pdf

O. Giménez and M. Noy, *Asymptotic Enumeration and Limit Laws of Planar Graphs* Journal of the American Mathematical Society, Vol. 22, Number 2, April 2009, pages 309–329.

E. A. Bender, Z. Gao and N.C. Wormald, *The number of labeled 2-connected planar graphs*. Electronic Journal of Combinatorics 9, 1 (2002), Research Paper 43, 13 pp.