Conference *Mathematics and random structures*
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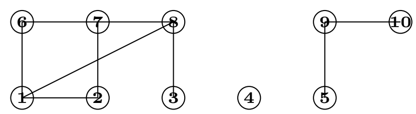
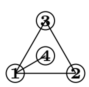
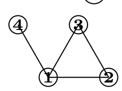
Enumeration of planar graphs: how symbolic method and singularity analysis apply

– Lecture notes –

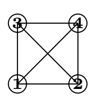
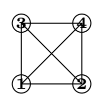
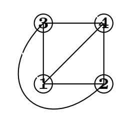
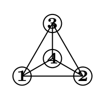
1 Introduction

Quick definition of a *graph* (in this talk, all graphs are labelled and simple: at most one edge between two vertices, thus no loops), characterized by its adjacence matrix.

Example (3 connected components): 

As labelled graphs,  and  are equal.

Quick definition of a *planar* graph: one can draw it in the plane without edge crossing.

Example: the graph  is not plane, but it is planar:  =  = 

The smallest nonplanar graph is the complete graph on 5 vertices (often named K_5).
Denote by g_n the number of planar graphs on n vertices.

Theorem 1 (Giménez and Noy, 2008) *When n tends to infinity,*

$$g_n \sim g n! \gamma^n n^{-7/2}$$

where $g \approx 0.42609.10^{-5}$ and $\gamma \approx 27.22688$ are explicit (but intricately defined) numbers.

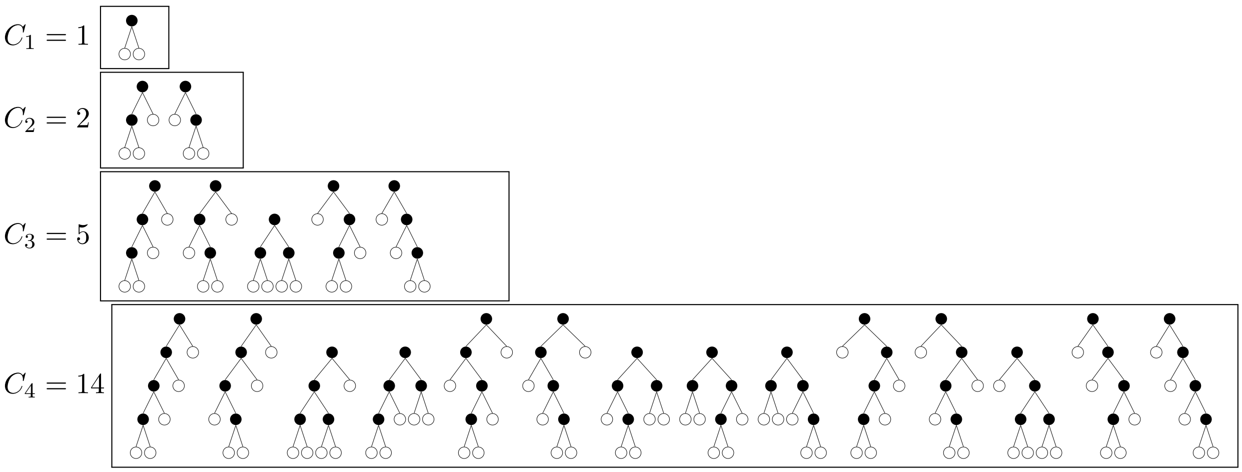
[Many consequences on random planar graphs and many other random combinatorial objects.]

How can one prove such a result?

This talk gives a rapid glance at a very powerful method for computing asymptotics of combinatorial objects: generating power series, symbolic method, analysis of singularities.

2 A very famous example: asymptotics of binary trees

Definition of a (rooted plane) binary tree, leaf, internal node.
Denote by C_n the number of (rooted plane binary) trees having n internal nodes.
Elementary counting when the number of vertices is small.



Consider the *generating (power) series* of binary trees

$$C(x) = \sum_{n\geq 0} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + \cdots$$

Combinatorial decomposition: a tree is either empty or is defined by a root with two subtrees (make a picture). Translating this fact as a recurrence relation on the C_n, and further into the generating series langage leads to the equation

$$C(x) = 1 + xC(x)^2. \tag{1}$$

This is a particular case of the powerful so-called *symbolic method*. Comments. See [FS] for a beautiful and rather complete landscape on the subject.

Solving quadratic Equation (1), one gets

$$C(x) = \frac{1-\sqrt{1-4x}}{2x},$$

the other solution being eliminated because C is a power series (nonnegative powers). Besides, Taylor series of the square root function asserts that

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} x^n,$$

the equality being valid on the open disc $D(0,1)$ for the principal determination of the square root (defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, values in $\{\Re z > 0\}$). This leads to the closed formula

$$C_n = -\frac{1}{2}(-4)^{n+1}\binom{1/2}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Using Stirling formula, and one gets the asymptotics when n tends to infinity:

$$C_n \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}.$$

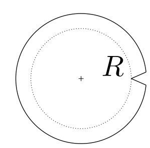
This result can be directly proven using the so-called *Transfer Theorem* that asserts the following (the proof belongs to complex analysis, there are many extensions of this theorem).

Theorem 2 (Transfer Theorem) *Let $S(x) = \sum s_n x^n$ be a power series with complex coefficients. Assume that:*
(i) S has $R > 0$ as a convergence radius
(ii) in a camembert domain around R, the function $x \mapsto S(x)$ is analytic and admits an expansion of the form

$$S(x) \sim C \left(1 - \frac{x}{R}\right)^{\alpha}$$

when x tends to R in the domain, with $C \in \mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$.
Then, when n tends to infinity,

$$s_n \sim \frac{C}{\Gamma(-\alpha)} R^{-n} n^{-\alpha-1}.$$

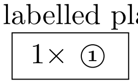
A *camembert domain around R* is an open subset of the complex plane of the form 

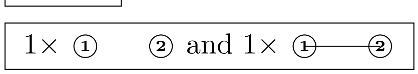
Evocation of singularity analysis, many many examples, see [FS].
Efficiency of coupling symbolic method and singularity analysis.

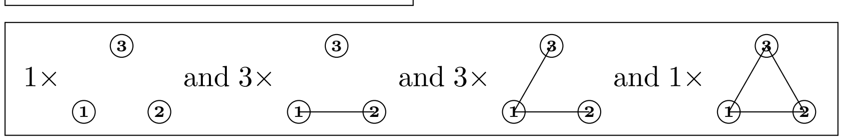
3 How symbolic method and singularity analysis apply to planar graphs

A graph has vertices and edges that join vertices. A graph is said *planar* is it can be drawn in the plane (on on the sphere S^2) without edge crossing. The graphs we consider here are simple (no loops and at most one edge between two vertices) and labelled (a label $1, 2, 3, \ldots$ is given to each vertex).
[Beware: graph $\neq$ map.]

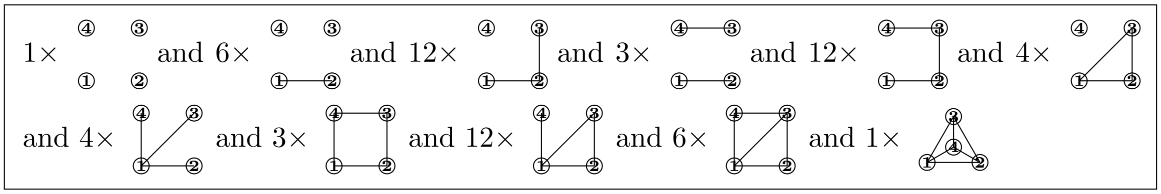
Small labelled planar graphs:

$g_1 = 1$ 

$g_2 = 2$ 

$g_3 = 8$ 

$g_4 = 64$



Consider the *exponential* generating series of planar graphs (Sloane's A066537)

$$G(x) = \sum_{n \geq 0} \frac{g_n}{n!} x^n = 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 64\frac{x^4}{4!} + 1023\frac{x^5}{5!} + 32071\frac{x^6}{6!} + 1823707\frac{x^7}{7!} \cdots$$

[Exponential generating series are suitable ones for labelled objects, see [FS].]
The proof of Theorem 1 is based on the analysis of singularities of G.

– What can be said on G so that one can make this singularity analysis ? –

Definition of a *connected* planar graph.
By symbolic method (a graph is a *set* of connected graphs),

$$G(x) = \exp C(x)$$

where

$$C(x) = \sum_{n \geq 0} \frac{c_n}{n!} x^n = 1 + x + \frac{x^2}{2!} + 4\frac{x^3}{3!} + 38\frac{x^4}{4!} + 727\frac{x^5}{5!} + 26013\frac{x^6}{6!} + 1597690\frac{x^7}{7!} \cdots$$

is the exponential generating series of connected planar graphs. Analytic singularities of G will be deduced from singularities of C.

Definition of a *2-connected* planar graph. Denote by

$$B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$$

the exponential generating series of 2-connected planar graphs. From the combinatoric decomposition of *pointed* connected planar graphs (a graph with a chosen vertex) in pointed 2-connected components, the symbolic method implies that

$$xC'(x) = x \exp\left[B'\left(xC'(x)\right)\right].$$

Thus, the power series $F(x) = xC'(x)$ satisfies the implicit equation

$$F(x) = x \exp B'\left(F(x)\right), \tag{2}$$

so that analytic singularities of F are related to those of B (or B').

Works from Tutte's sphere, Mullin and Schellenberg (1968), Walsh (1982), Bender, Gao and Wormald ([BGW], 2002) on maps and graphs show the following successive facts. They are all due to combinatorial properties of graphs and maps, translated into the generating function langage. Denote by

$$B_2(x,y) = \sum_{n\geq 0, q\geq 0} b_{n,q} y^q \frac{x^n}{n!}$$

the *bivariate* power series in x and y, where $b_{n,q}$ denotes the number of 2-connected planar graphs having n (labelled) vertices and q (non labelled) edges. Note that

$$B(x) = B_2(x,1).$$

Claim *There are bivariate power series $U(x,z)$, $D(x,y)$ and $M(x,y)$ related by the following relations:*

$$U(x,y) = xy\left(1 + y\left(1 + U(x,y)\right)^2\right)^2 \tag{3}$$

$$M(x,y) = Rat\left(x, y, U(x,y)\right) \tag{4}$$

$$\frac{M\left(x, D(x,y)\right)}{2x^2 D(x,y)} - \log\frac{1 + D(x,y)}{1+y} + \frac{xD^2(x,y)}{1 + xD(x,y)} = 0 \tag{5}$$

$$\frac{\partial B_2(x,y)}{\partial y} = \frac{x^2}{2}\left[\frac{1 + D(x,y)}{1+y} - 1\right]. \tag{6}$$

In Formula (4), Rat denotes an explicit simple rational fraction in 3 variables on $\mathbb{Q}$ which can be easily written on one line (see [BGW] or [GN]). In fact, D and M have combinatorical interpretations: M is the bivariate generating series of 3-connected rooted maps and D is the bivariate generating series of planar graphs with two distinguished vertices, called *poles*, such that adding an edge between the poles creates a 2-connected planar graph.
From (6), (5), (3) and (4), one deduces first that B has a unique dominant singularity at some positive number R and then that it admits an expansion of the form

$$B(x) = B_0 + B_2\left(1 - \frac{x}{R}\right) + B_4\left(1 - \frac{x}{R}\right)^2 + B_5\left(1 - \frac{x}{R}\right)^{\frac{5}{2}} + O\left(1 - \frac{x}{R}\right)^3$$

when x tends to R in a suitable camembert domain. The positive real number R is defined by some transcendental equation of the form $h(R) = 0$, the function h being explicitely written in the rational-exponential algebra. The numbers B_k are log-rational explicit functions of R.
Coming back to Relation (2), one shows that F admits a unique dominant singularity at $\rho = Re^{B_2/R}$ together with an expansion of the form

$$F(x) = F_0 + F_2\left(1 - \frac{x}{\rho}\right) + F_3\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}} + O\left(1 - \frac{x}{\rho}\right)^2$$

when x tends to ρ in a suitable camembert domain. The numbers F_k are explicit functions of R.
Finally, Giménez and Noy show that G admits a unique singularity at ρ and an expansion at ρ in a camembert domain of the form

$$G(x) = G_0 + G_2\left(1 - \frac{x}{\rho}\right) + G_4\left(1 - \frac{x}{\rho}\right)^2 + G_5\left(1 - \frac{x}{\rho}\right)^{\frac{5}{2}} + O\left(1 - \frac{x}{\rho}\right)^3,$$

where the G_k are explicit functions of R and the B_k. The last step consists in using the transfer lemma that leads to Theorem 1.

4 A very short bibliography

[BGW] E. A. Bender, Z. Gao and N.C. Wormald, *The number of labeled 2-connected planar graphs.* Electronic Journal of Combinatorics 9, 1 (2002), Research Paper 43, 13 pp.

[FS] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009. Free acces at `http://algo.inria.fr/flajolet/Publications/book.pdf`

[GN] O. Giménez and M. Noy, *Asymptotic Enumeration and Limit Laws of Planar Graphs* Journal of the American Mathematical Society, Vol. 22, Number 2, April 2009, pages 309–329.