1 Introduction

Quick definition of a graph (in this talk, all graphs are labelled and simple: at most one edge between two vertices, thus no loops), characterized by its adjacency matrix.

Example (3 connected components):

\[
\begin{array}{c}
\text{As labelled graphs, } \quad \begin{array}{c}
\text{and } \\
\end{array}
\end{array}
\]

Quick definition of a planar graph: one can draw it in the plane without edge crossing.

Example: the graph \( \begin{array}{c}
\text{is not plane, but it is planar: } \\
\end{array} \) \( \begin{array}{c}
\text{= } \\
\text{= } \\
\end{array} \)

The smallest nonplanar graph is the complete graph on 5 vertices (often named \( K_5 \)).

Denote by \( g_n \) the number of planar graphs on \( n \) vertices.

Theorem 1 (Giménez and Noy, 2008)  When \( n \) tends to infinity,

\[
g_n \sim g_n ! \gamma^n n^{-7/2}
\]

where \( g \approx 0.42609.10^{-5} \) and \( \gamma \approx 27.22688 \) are explicit (but intricately defined) numbers.

[Many consequences on random planar graphs and many other random combinatorial objects.]

How can one prove such a result?

This talk gives a rapid glance at a very powerful method for computing asymptotics of combinatorial objects: generating power series, symbolic method, analysis of singularities.
2 A very famous example: asymptotics of binary trees

Definition of a (rooted plane) binary tree, leaf, internal node.
Denote by $C_n$ the number of (rooted plane binary) trees having $n$ internal nodes.
Elementary counting when the number of vertices is small.

$C_1 = 1$

$C_2 = 2$

$C_3 = 5$

$C_4 = 14$

Consider the generating (power) series of binary trees

$$C(x) = \sum_{n \geq 0} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + \cdots$$

Combinatorial decomposition: a tree is either empty or is defined by a root with two subtrees (make a picture). Translating this fact as a recurrence relation on the $C_n$, and further into the generating series language leads to the equation

$$C(x) = 1 + xC(x)^2. \quad (1)$$

This is a particular case of the powerful so-called symbolic method. Comments. See [FS] for a beautiful and rather complete landscape on the subject.

Solving quadratic Equation (1), one gets

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

the other solution being eliminated because $C$ is a power series (nonnegative powers). Besides, Taylor series of the square root function asserts that

$$\sqrt{1 + x} = (1 + x)^{\frac{1}{2}} = \sum_{n \geq 0} \left(\frac{1}{2}\right)_n x^n,$$

the equality being valid on the open disc $D(0, 1)$ for the principal determination of the square root (defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, values in $\{\Re z > 0\}$). This leads to the closed formula

$$C_n = \frac{1}{2} (-4)^{n+1} \left(\frac{1}{2}\right)_n \binom{n+1}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$
Using Stirling formula, and one gets the asymptotics when \( n \) tends to infinity:

\[
C_n \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}.
\]

This result can be directly proven using the so-called \textit{Transfer Theorem} that asserts the following (the proof belongs to complex analysis, there are many extensions of this theorem).

\textbf{Theorem 2 (Transfer Theorem)} Let \( S(x) = \sum s_n x^n \) be a power series with complex coefficients. Assume that:

(i) \( S \) has \( R > 0 \) as a convergence radius

(ii) in a camembert domain around \( R \), the function \( x \mapsto S(x) \) is analytic and admits an expansion of the form

\[
S(x) \sim C \left(1 - \frac{x}{R}\right)^\alpha
\]

when \( x \) tends to \( R \) in the domain, with \( C \in \mathbb{C} \) and \( \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \).

Then, when \( n \) tends to infinity,

\[
s_n \sim \frac{C}{\Gamma(-\alpha)} R^{-n} n^{-\alpha-1}.
\]

A \textit{camembert domain around} \( R \) is an open subset of the complex plane of the form

\[
\text{A camembert domain around } R \text{ is an open subset of the complex plane of the form } \bigcirc \text{. } R
\]

Evocation of singularity analysis, many many examples, see [FS].

Efficiency of coupling symbolic method and singularity analysis.

\section*{3 \ How symbolic method and singularity analysis apply to planar graphs}

A graph has vertices and edges that join vertices. A graph is said \textit{planar} is it can be drawn in the plane (on on the sphere \( S^2 \)) without edge crossing. The graphs we consider here are simple (no loops and at most one edge between two vertices) and labelled (a label 1, 2, 3, \ldots is given to each vertex).

[Beware: graph \( \neq \) map.]

Small labelled planar graphs:

\begin{itemize}
  \item \( g_1 = 1 \) \hspace{1cm} \begin{array}{c}
  1 \times \begin{array}{c}
    1
  \end{array}
  \end{array}
  \\
  \item \( g_2 = 2 \) \hspace{1cm} \begin{array}{c}
  1 \times \begin{array}{c}
    1
  \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      2
    \end{array}
  \end{array}
  \text{ and } \begin{array}{c}
  1 \times \begin{array}{c}
    1
  \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      4
    \end{array}
  \end{array}
  \\
  \item \( g_3 = 8 \) \hspace{1cm} \begin{array}{c}
  1 \times \begin{array}{c}
    1
  \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      2
    \end{array}
  \end{array}
  \text{ and } \begin{array}{c}
  3 \times \begin{array}{c}
    3
  \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      3
    \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      \begin{array}{c}
        2
      \end{array}
    \end{array}
  \end{array}
  \text{ and } \begin{array}{c}
  1 \times \begin{array}{c}
    1
  \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      2
    \end{array}
  \end{array}
  \end{array}
  \end{array}
\end{itemize}
Consider the exponential generating series of planar graphs (Sloane’s A066537)

\[
G(x) = \sum_{n \geq 0} \frac{g_n x^n}{n!} = 1 + x + \frac{2}{2!} x^2 + \frac{8}{3!} x^3 + \frac{64}{4!} x^4 + \frac{1023}{5!} x^5 + \frac{32071}{6!} x^6 + \frac{1823707}{7!} x^7 \cdots
\]

[Exponential generating series are suitable ones for labelled objects, see [FS].] The proof of Theorem 1 is based on the analysis of singularities of \(G\).

– What can be said on \(G\) so that one can make this singularity analysis ? –

Definition of a connected planar graph. By symbolic method (a graph is a set of connected graphs),

\[
G(x) = \exp C(x)
\]

where

\[
C(x) = \sum_{n \geq 0} \frac{c_n x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{4}{3!} x^3 + \frac{38}{4!} x^4 + \frac{727}{5!} x^5 + \frac{26013}{6!} x^6 + \frac{1597690}{7!} x^7 \cdots
\]

is the exponential generating series of connected planar graphs. Analytic singularities of \(G\) will be deduced from singularities of \(C\).

Definition of a 2-connected planar graph. Denote by

\[
B(x) = \sum_{n \geq 0} \frac{b_n x^n}{n!}
\]

the exponential generating series of 2-connected planar graphs. From the combinatoric decomposition of pointed connected planar graphs (a graph with a chosen vertex) in pointed 2-connected components, the symbolic method implies that

\[
x C'(x) = x \exp \left[ B'(xC'(x)) \right].
\]

Thus, the power series \(F(x) = x C'(x)\) satisfies the implicit equation

\[
F(x) = x \exp B'(F(x)), \tag{2}
\]

so that analytic singularities of \(F\) are related to those of \(B\) (or \(B'\)).

Works from Tutte’s sphere, Mullin and Schellenberg (1968), Walsh (1982), Bender, Gao and Wormald ([BGW], 2002) on maps and graphs show the following successive facts. They are
all due to combinatorial properties of graphs and maps, translated into the generating function language. Denote by

\[ B_2(x, y) = \sum_{n \geq 0, q \geq 0} b_{n,q} x^n y^q n! \]

the bivariate power series in \( x \) and \( y \), where \( b_{n,q} \) denotes the number of 2-connected planar graphs having \( n \) (labelled) vertices and \( q \) (non labelled) edges. Note that

\[ B(x) = B_2(x, 1). \]

**Claim** There are bivariate power series \( U(x, z) \), \( D(x, y) \) and \( M(x, y) \) related by the following relations:

\[ U(x, y) = xy \left( 1 + y (1 + U(x, y))^2 \right)^2 \]  (3)
\[ M(x, y) = \text{Rat}(x, y, U(x, y)) \]  (4)
\[ \frac{M(x, D(x, y))}{2x^2 D(x, y)} - \log \frac{1 + D(x, y)}{1 + y} + \frac{x D^2(x, y)}{1 + x D(x, y)} = 0 \]  (5)
\[ \frac{\partial B_2(x, y)}{\partial y} = \frac{x^2}{2} \left[ \frac{1 + D(x, y)}{1 + y} - 1 \right]. \]  (6)

In Formula (4), \( \text{Rat} \) denotes an explicit simple rational fraction in 3 variables on \( \mathbb{Q} \) which can be easily written on one line (see [BGW] or [GN]). In fact, \( D \) and \( M \) have combinatorial interpretations: \( M \) is the bivariate generating series of 3-connected rooted maps and \( D \) is the bivariate generating series of planar graphs with two distinguished vertices, called poles, such that adding an edge between the poles creates a 2-connected planar graph.

From (6), (5), (3) and (4), one deduces first that \( B \) has a unique dominant singularity at some positive number \( R \) and then that it admits an expansion of the form

\[ B(x) = B_0 + B_2 \left( 1 - \frac{x}{R} \right) + B_4 \left( 1 - \frac{x}{R} \right)^2 + B_5 \left( 1 - \frac{x}{R} \right)^{\frac{5}{2}} + O \left( 1 - \frac{x}{R} \right)^3 \]

when \( x \) tends to \( R \) in a suitable camembert domain. The positive real number \( R \) is defined by some transcendental equation of the form \( h(R) = 0 \), the function \( h \) being explicitly written in the rational-exponential algebra. The numbers \( B_k \) are log-rational explicit functions of \( R \).

Coming back to Relation (2), one shows that \( F \) admits a unique dominant singularity at \( \rho = Re^{B_2/R} \) together with an expansion of the form

\[ F(x) = F_0 + F_2 \left( 1 - \frac{x}{\rho} \right) + F_3 \left( 1 - \frac{x}{\rho} \right)^{\frac{3}{2}} + O \left( 1 - \frac{x}{\rho} \right)^2 \]

when \( x \) tends to \( \rho \) in a suitable camembert domain. The numbers \( F_k \) are explicit functions of \( R \).

Finally, Giménez and Noy show that \( G \) admits a unique singularity at \( \rho \) and an expansion at \( \rho \) in a camembert domain of the form

\[ G(x) = G_0 + G_2 \left( 1 - \frac{x}{\rho} \right) + G_4 \left( 1 - \frac{x}{\rho} \right)^2 + G_5 \left( 1 - \frac{x}{\rho} \right)^{\frac{5}{2}} + O \left( 1 - \frac{x}{\rho} \right)^3, \]

where the \( G_k \) are explicit functions of \( R \) and the \( B_k \). The last step consists in using the transfer lemma that leads to Theorem I.
4 A very short bibliography

